ON SOLUTIONS TO COURNOT-NASH EQUILIBRIA EQUATIONS ON THE SPHERE

MICAH WARREN

1. Introduction

In this note, we discuss equations associated to Cournot-Nash Equilibria as put forward by Blanchet and Carlier [1]. These equations are related to an optimal transport problem in which the source measure is known but the target measure is to be determined. A Cournot-Nash Equilibrium is a special type of optimal transport: Each individual x is transported to a point $T(x)$ in a way that not only minimizes the total cost of transportation, but minimizes a cost to the indivual x (transportation plus other). This latter cost may depend on the target distribution, and may involve congestion, isolation and geographical terms.

Blanchet and Carlier demonstrated how CNE are related to nonlinear elliptic PDEs, explicitly deriving a Euclidean version of the equation [1, eq 4.6] and showing [1, Theorem 3.8] that this problem has some very nice properties. The fully nonlinear Monge-Ampère equation differs from 'standard' optimal transport equations in that the potential itself occurs on the right hand side, along with possibly some nonlocal terms. Here we study the problem on the sphere. Immediately one can conclude from [1, Theorem 3.8] and Loeper's [8] results that that optimal maps are continuous with control on the Hölder norm. We move this a step further and show that all derivative norms can be controlled in terms of the data, when the solution is smooth. When the solution is known to be differentiable enough, then one can easily adapt Ma-Trudinger-Wang's [9] estimates. To make the conclusion a priori, we must use the continuity method. Closedness follows Ma-Trudinger-Wang's estimates, but openness is not immediate and requires some conditions. In Theorem 6 we give some conditions on the data so that the problem can be solved smoothly.

2. Background and setup

In this section we briefly recap the setup in [1]. Given a space of player types X, endowed with a probability measure μ, an action space Y, and a cost function

$$\Phi : X \times Y \times \mathcal{P}(Y) \rightarrow \mathbb{R},$$

x-type agents pay cost $\Phi(x, y, \nu)$ to take action y. Here $\nu \in \mathcal{P}(Y)$ is the probability measure in the action space which is the push forward of μ via by the map of actions from X to Y. Supposing that x-type agents know the distribution ν, they can choose

I'm supported in part by NSF DMS-0901644 and NSF DMS-1161498. Thanks also to my representative Rush Holt for standing up to defend scientific funding.

the best action y. A Cournot-Nash Equilibrium is a joint probability distribution measure $\gamma \in \mathcal{P}(X \times Y)$ with first marginal μ such that

$$\gamma \left\{ (x, y) \in X \times Y : \Phi(x, y, \nu) = \min_{z \in Y} \Phi(x, z, \nu) \right\} = 1 \tag{2.1}$$

where ν is the second marginal.

We will be interested in a particular type of cost

$$\Phi(x, y, \nu) = c(x, y) + \mathcal{V}[\nu](y)$$

where c is the transportation cost. Blanchet and Carlier show [1, Lemma 2.2] that a CNE will necessarily be an optimal transport pairing for the cost c between the measures μ and ν. They further show that if $\mathcal{V}[\nu]$ is the differential of a functional $\mathcal{E}[\nu]$, then at a minimizer for $\mathcal{E}[\nu] + \mathcal{W}_c(\mu, \nu)$, the optimal transport will necessarily be a CNE. (Here, $\mathcal{W}_c(\mu, \nu)$ is Wasserstein distance.) In particular, if the cost $\mathcal{V}_m[\nu]$ is of the form

$$\mathcal{V}_m[\nu](y) = f\left(\frac{d\nu}{dm}(y)\right) + \int \phi(y, z) d\nu(z) + V(y) \tag{2.2}$$

where m is a 'background' measure and the function $\phi(y, z)$ is symmetric on $Y \times Y$, then $\mathcal{V}_m$ is a differential and solution to the optimal transport is a CNE. (We will be licentious with notation, letting ν denote not only the measure, but also the density with respect to the background m.) From here on out we suppose we are working with a solution to an optimal transport with cost c between measures μ and ν which is also a CNE for a total cost Φ. We also assume that the manifolds X and Y are compact without boundary.

One can consider the pair (u, u^*) which maximizes the Kantorovich functional

$$J(u, v) = \int -u d\mu + \int v d\nu$$

over all $-u(x) + v(y) \leq \Phi(x, y)$. The pair (u, u^*) will satisfy

$$-u(x) + u^*(y) = \Phi(x, y) \tag{2.3}$$

γ- almost everywhere, where γ is the optimal measure for the Kantorovich problem. If the cost satisfies the standard Spence-Mirrlees condition (in the mathematics literature, the "twist" or (A1) condition (c.f. [9, section 2])), we have μ-almost everywhere

$$-u(x) + u^*(T(x)) = \Phi(x, T(x)). \tag{2.4}$$

The twist condition says that $T(x)$ is uniquely determined by

$$T(x) = \{y : Du(x) + Dc(x, y) = 0\}, \tag{2.5}$$

which gives the identity

$$Du(x) + Dc(x, T(x)) = 0. \tag{2.6}$$

Note that fixing an x, the quantity

$$\Phi(x, y) - u^*(y)$$

must have a minimum at $T(x)$, we conclude that

$$D_y \Phi(x, T(x)) = Du^*(T(x)).$$

But then we bring in the condition (2.1) that, fixing x,

$$\Phi(x, T(x)) \leq \Phi(x, y)$$

which implies that

$$D_y \Phi(x, T(x)) = 0$$

from which we conclude that

$$Du^*(y) \equiv 0.$$

Now the pair (u, u^*) is determined up to a constant. One can choose the constant in u or u^* but not both. At this point we simply choose $u^* = 0$. Having fixed this choice allows us to read information about u and the measure ν, using (2.2) and (2.4)

$$-u(x) = c(x, T(x)) + f(\nu(T(x))) + \int \phi(T(x), z) d\nu(z) + V(T(x)).$$

In particular, the density $\nu(y)$ must be determined by

(2.7)
$$\nu(T(x)) = f^{-1}\left\{-u(x) - c(x, T(x)) - \int \phi(T(x), T(z)) d\mu(z)(z) - V(T(x))\right\}$$

having used the change of integration variables T between μ and ν. The optimal transportation equation (c.f. [9]) becomes

$$\frac{\det(u_{ij}(x) + c_{ij}(x, T(x)))}{\det\left(-c_{is}(x, T(x))\right)} = \frac{\mu(x)}{f^{-1}\left\{Q(x, u)\right\}}. \tag{2.8}$$

Here and in the sequel, we use i, j, k to denote derivatives in the source X, and p, s, t to denote derivatives in the target Y. It will convenient to assume that c_{is} is negative definite, which follows if we are assuming the A2 condition (see [9]) and have chosen an appropriate coordinate system). We will use $b_{is}(x) = -c_{is}(x, T(x))$. Also (to keep equations within one line) we abbreviate

$$Q(x, u) = -u(x) - c(x, T(x)) - \int \phi(T(x), T(z)) d\mu(z) - V(T(x))$$

with $T(x)$ being determined by (2.5).

Before we say how this fully nonlinear equation is vulnerable, we mention the "Inada-like" conditions [1, Section 3.3] :

$$\lim_{\nu \to 0^+} f(\nu) = -\infty \quad \text{and} \quad \lim_{\nu \to +\infty} f(\nu) = +\infty \tag{2.9}$$

$$f' > 0 \quad \text{and} \quad f \in C^2(\mathbb{R}^+). \tag{2.10}$$

If f satisfies these conditions, then several observations are in order. First as noted in [1, Theorem 3.8] on a compact manifold we get bounds away from zero and infinity for the density ν. In the spherical distance squared transportation cost case, this immediately gives C^α continuity of the map by results of Loeper. Secondly, the right hand side of the equation (2.8) is strictly monotone in the zeroth order term - this is crucial in obtaining existence and uniqueness results, as it will allow us to invert the linearized operator. Finally, as we will show below, the first derivatives of this density will be bounded in terms of an a priori constant (depending on smoothness of f) and the second derivatives will be bounded by a constant times second derivatives of u. These estimates will allow us to take advantage of the Ma-Trudinger-Wang estimates.

We will show an estimate on smooth solutions: If a solution to (2.8) is C^4, then it enjoys estimates of all orders subject to universal bounds. In order to show that arbitrary solutions are C^4 and hence smooth, we must use a continuity method. This method relies on a linearization which requires some discussion, given the integral terms in the equation.

The problem here, on a compact manifold, with cost function satisfying the Ma-Trudinger-Wang condition, is quite a bit simpler than the more delicate boundary value problem mentioned in [1]. With or without the nonlocal terms, such a problem may be approached as in [7]. We leave this problem aside for now.

3. Linearization

We take the natural log of (2.8) and then consider the functional

$$F(x,u,Du,D^2u) = \ln\det((u_{ij}(x) + c_{ij}(x,T(x))) - \ln\det\left(b_{is}(x,T(x))\right) \\ - \ln\mu(x) + \ln f^{-1}\left(Q(x,u)\right)) \tag{3.1}$$

and the equation we want to solve is

$$F(x,u,Du,D^2u) = 0. \tag{3.2}$$

Preparing for linearization, consider (2.6) applied to $u+tv$:

$$Du(x) + tD\eta(x) + Dc(x,T_t(x)) = 0.$$

Differentiate with respect to t and get that

$$D\eta(x) = b_{is}(x,T(x))\frac{dT^s}{dt}.$$

Linearizing,

(3.3)

$$L\eta = \frac{d}{dt}F(u+t\eta) = w^{ij}\eta_{ij} + w^{ij}c_{ijs}b^{sk}\eta_k + b^{is}c_{isp}b^{pk}\eta_k$$

(3.4)

$$+\frac{\left(f^{-1}(Q)\right)'}{f^{-1}\left(Q\right)}\left\{\begin{array}{c} -c_s(x,T(x))b^{sk}\eta_k - \eta - V_s b^{sk}\eta_k \\ -b^{sk}\eta_k(x)\int\phi_s(T(x),T(z))d\mu(z) - \int\phi_{\bar{s}}(T(x),T(z))b^{sk}(z)\eta_k(z)d\mu(z)\end{array}\right\}.$$

Here we are using

$$w_{ij}(x) = u_{ij}(x) + c_{ij}(x,T(x)).$$

We note also that differentiating (2.6) shows

$$T_i^s(x,T(x)) = \frac{\partial T^s}{\partial x_i} = b^{sk}(x,T(x))w_{ki}(x,T(x)). \tag{3.5}$$

Splitting (3.3) and (3.4) for convenience we write, respectively,

$$L\eta = L^0\eta + L^1\eta.$$

We take $g_{ij}(x) = w_{ij}(x)$ to define a metric (one can check that it transforms as such), then write

$$d\mu(x) = e^{-a(x)}dV_g(x) \tag{3.6}$$

where

$$-a(x) = \ln\mu(x) - \frac{1}{2}\ln\det w_{ij}(x).$$

From the definition of F (3.1) we have

$$-a(x)=\frac{1}{2}\ln\det w_{ij}-\ln\det b+\ln\nu-F,$$

having introduced

$$\nu(x)=\ln f^{-1}\left(Q(x,u)\right).$$

First, we compute the Bakry-Emery Laplace

$$\triangle_a\eta=\triangle_g\eta-\nabla a\cdot\nabla\eta.$$

We begin with $\triangle_g\eta$ differentiating in some coordinate system (see very similar computations preceding 4.1):

$$\frac{1}{\sqrt{\det w}}\left(\sqrt{\det w}w^{ij}\eta_j\right)_i=w^{ij}\eta_{ij}+\frac{1}{2}w^{ab}\partial_i w_{ab}w^{ij}\eta_j-w^{ia}w^{bj}\partial_i w_{ab}\eta_j$$

$$\begin{aligned}
&=w^{ij}\eta_{ij}+w^{ab}w^{ij}\left(\partial_i w_{ab}-\partial_b w_{ia}\right)\eta_j-\frac{1}{2}w^{ab}\partial_i w_{ab}w^{ij}\eta_j\\
&=w^{ij}\eta_{ij}+\left(w^{ba}c_{abs}b^{sj}-w^{ij}c_{isk}b^{sk}\right)\eta_j-\frac{1}{2}w^{ij}\left(\ln\det w\right)_i\eta_j\\
&=L^0\eta-b^{is}c_{isp}b^{pk}\eta_k-w^{ij}c_{kis}b^{sk}\eta_j-\frac{1}{2}w^{ij}\left(\ln\det w\right)_i\eta_j.
\end{aligned}$$

Thus

$$\begin{aligned}
\triangle_a\eta&=L^0\eta-b^{is}c_{isp}b^{pk}\eta_k-w^{ij}c_{kis}b^{sk}\eta_j-\frac{1}{2}w^{ij}\left(\ln\det w\right)_i\eta_j\\
&+\frac{1}{2}w^{ij}\left(\ln\det w\right)_i\eta_j-w^{ij}\left(\ln\det b\right)_i\eta_j+\left(\ln\nu\right)_i w^{ij}\eta_j-F_i w^{ij}\eta_j\\
&=L^0 v+\left(\ln\nu\right)_i w^{ij}\eta_j-F_i w^{ij}\eta_j,
\end{aligned}$$

and hence

$$L\eta=\triangle_a\eta+L^1\eta-(\ln\nu)_i w^{ij}\eta_j+F_i w^{ij}\eta_j.$$

Next, we compute

$$(\ln\nu)_i=\frac{\left(f^{-1}(Q)\right)'}{f^{-1}(Q)}\left\{\begin{array}{c}-u_i(x)-c_i(x,T(x))-c_s(x,T(x))b^{sk}w_{ki}\\-b^{sk}w_{ki}\int\phi_s(T(x),T(z))d\mu(z)-V_s b^{sk}w_{ki}\end{array}\right\}.$$

Noting that $-u_i(x)-c_i(x,T(x))=0$, and the expression (3.4) we have

$$L^1\eta-(\ln\nu)_i w^{ij}\eta_j=$$

$$\frac{\left(f^{-1}(Q)\right)'}{f^{-1}(Q)}\left\{-\eta-\int\phi_s(T(x),T(z))b^{sk}(z)\eta_k(z)d\mu(z)\right\}.$$

Next, we compute the integral term in the previous expression: Notice

$$\int\left\langle\nabla\phi(y,T(z)),\nabla\eta\right\rangle e^{-a(z)}dV_g(z)=\int\phi_s(y,T(z))b^{sk}w_{ki}\eta_j w^{ij}e^{-a(x)}dV_g$$

$$=\int\phi_s(T(x),T(z))b^{sk}\eta_k(z)d\mu(z).$$

Now, integrating by parts, we have that

$$-\int\phi_s(T(x),T(z))b^{sk}\eta_k(z)d\mu(z)=\int\phi(T(x),T(z))\triangle_a\eta(z)e^{-a(z)}dV_g(z).$$

Combining, we have

$$L\eta = \triangle_a \eta - h(x)\eta(x) - h(x)\int \phi(T(x),T(z)) \triangle_a \eta(z) d\mu(z) + \langle \nabla F, \nabla \eta \rangle, \tag{3.7}$$

using the shorthand

$$h(x,u) = \frac{\left(f^{-1}(Q)\right)'}{f^{-1}(Q)}.$$

Note here that if f satisfies (2.9),(2.10) then $h(x,u)$ will be a positive differentiable quantity. In particular, if $f(\tau) = ln(\tau)$ then h will be identically 1. When $F \equiv 0$ we have the following.

Proposition 1. *At a solution of (3.2), the linearized operator takes the form*

$$L\eta = \triangle_a \eta - h(x)\eta(x) - h(x)\int \phi(T(x),T(z)) \triangle_a \eta(z) d\mu(z). \tag{3.8}$$

Lemma 2. *Suppose that*

$$\max_{(x,y)\in X\times Y} h(x,u)|\phi(x,y)| < 1. \tag{3.9}$$

Then the operator (3.8) has trivial kernel.

Proof. To make use of some functional analytic formality, we define operators A, J, h and I on the space

$$\mathcal{B} = L^2(X, d\mu)$$

via

$$\begin{aligned} [A\eta](x) &= \triangle_a \eta(x), \\ [J\eta](x) &= \int \phi(T(x),T(z))\eta(z)d\mu(z), \\ [h\eta](x) &= h(x)\eta(x) \\ [I\eta](x) &= \eta(x). \end{aligned}$$

Then

$$L = A - h - hJA = (I - hJ)A - h = (I - hJ)\left(A - (I - hJ)^{-1} h\right).$$

First we have the pointwise estimate

$$\begin{aligned} [hJ\eta](x) &= \int h(x)\phi(T(x),T(y))\eta(y)d\mu(y) \\ &\leq \left\| \int h(x)\phi(T(x),T(y))d\mu(x) \right\|_{L^2}^{1/2} \|\eta\|_{L^2}^{1/2} \\ &\leq \left[\max_{x,y\in X\times Y} h(x)|\phi(x,y)|\right]^{1/2} < \|\eta\|_{L^2}^{1/2} \end{aligned}$$

using (3.9). Integrating this quantity over μ yields that

$$\|hJ\| < 1$$

as an operator on $\mathcal{B}$. It then makes sense to talk about $(I - hJ)^{-1}$. Thus

$$Ker(L) = Ker\left(A - (I - hJ)^{-1} h\right).$$

Now suppose for purposes of contradiction, that we have nontrivial $\eta \in Ker(L)$. Then

$$A\eta = (I - hJ)^{-1} h\eta$$

thus

$$\left\langle (I - hJ)^{-1} h\eta, \eta \right\rangle = \langle A\eta, \eta\rangle = -\int |\nabla \eta|^2 \, d\mu < 0.$$

But as $(I - hJ)$ is invertible we can let

$$(I - hJ)\,\omega = h\eta$$

that is

$$\left\langle \omega, h^{-1} (I - hJ)\,\omega \right\rangle = \left\langle (I - hJ)^{-1} h\eta, \eta \right\rangle < 0$$

that is

$$0 > \left\langle \omega, \frac{1}{h}\omega \right\rangle - \langle \omega, J\omega\rangle \geq \frac{1}{\max h} \|\omega\|^2 - \|J\| \, \|\omega\|^2 = \left(\frac{1}{\max h} - \|J\| \right) \|\omega\|^2$$

which is clearly a contradiction if $1 > \max h \, \|J\|$. □

4. Estimates on the sphere

From here out we specialize to the round unit sphere, with cost function half of distance squared. Note that this sphere has Riemannian volume $n\omega_n$.

Oscillation estimates. The following estimates are a version of [1, Lemma 3.7]. On a compact manifold, the cost function will be bounded. Since the solution u is c-convex, at the maximum point x_{max} of u, u is supported below by cost support function $c(x, T(x_0)) + \lambda$. Hence, at the minimum point x_{min} we have that $u(x_{min}) \geq c(x_{min}, T(x_{max})) + \lambda$, which in turn tells us that

$$\operatorname{osc} u \leq \operatorname{osc} c = \frac{\pi^2}{2}.$$

Next we observe that, because integration of the density ν against m gives a probability measure, the density ν must be larger than $1/n\omega_n$ at some point y_0. It follows that, at the point $x_0 = T^{-1}(y_0)$ using (2.7)

$$-c(x_0, y_0) - u(x_0) - \int \phi(y_0, T(z)) d\mu(z) - V(y_0) \geq f(\frac{1}{n\omega_n})$$

and similarly at the point where the density ν is smallest, x_1

$$-c(x_1, y_1) - u(x_1) - \int \phi(y_1, T(z)) d\mu(z) - V(y_1) = f(\nu(x_1))$$

Hence,

$$-c(x_0, y_0) + c(x_1, y_1) - u(x_0) + u(x_1) - \int \left(\phi(y_0, T(z)) + \phi(y_1, T(z))\right) d\mu(z) - V(y_0) + V(y_1)$$
$$\geq f(\frac{1}{n\omega_n}) - f(\nu(x_1))$$

that is

$$f(\nu(x_1)) \geq f(\frac{1}{n\omega_n}) - 2\operatorname{osc} c - 2\operatorname{osc} \phi - \operatorname{osc} V > -\infty.$$

By Inada's condition,

$$\nu \geq f^{-1}\left(f(\frac{1}{n\omega_n}) - \pi^2 - 2\operatorname{osc} \phi - \operatorname{osc} V \right) > 0.$$

Similarly, an upper bound can be derived

$$\nu \le f^{-1}\left(f(\frac{1}{n\omega_n}) + \pi^2 + 2\mathrm{osc}\,\phi + \mathrm{osc}\,V\right) < \infty.$$

4.1. **Stayaway.** Now that ν is under control, it follows from the stayaway estimates of Delanoë and Loeper [2] that the map $T(x)$ must satisfy

$$dist_{\mathbb{S}^n}(x, T(x)) \le \pi - \epsilon(f, \mu, V, \phi)$$

In particular the map stays clear of the cut locus. All derivatives of the cost function are now controlled.

MTW estimates.

Lemma 3. *If the map T is differentiable and locally invertible, then the target measure density*

$$\nu(T(x)) = f^{-1}\left(-c(x, T(x)) - u(x) - \int \phi(T(x), T(z))d\mu(z) - V(T(x))\right)$$

has first derivatives bounded by a universal constant and has second derivatives bounded as

$$\nu_{sr} = C_1 + C_{2k}\left(T^{-1}\right)^k_r$$

where the constants are within a controlled range.

Proof. Differentiate in the x_k direction

$$\nu_s T^s_k(x) =$$

$$\left(f^{-1}\right)\prime\left\{-c_k(x, T(x)) - c_s(x, T(x))T^s_k - u_k - T^s_k \int \phi_s(T(x), T(z))d\mu(z) - V_s T^s_k\right\}$$

$$= \left(f^{-1}\right)\prime T^s_k(x)\left\{-c_s(x, T(x)) - \int \phi_s(T(x), T(z))d\mu(z) - V_s(T(x))\right\}.$$

As this is true for all k and DT is invertible, we can conclude that

$$\nu_s(T(x)) = \left(f^{-1}\right)\prime\left\{-c_s(x, T(x)) - \int \phi_s(T(x), T(z))d\mu(z) - V_s(T(x))\right\},$$

which is a bounded quantity. For second derivatives, differentiate this equation in x again

$$\nu_{sp}T^p_k =$$

$$\left(f^{-1}\right)\prime\prime T^p_k(x)\times$$

$$\left\{-c_s(x, T(x)) - \int \phi_s(T(x), T(z))d\mu(z) - V_s(T(x))\right\}$$

$$\times\left\{-c_p(x, T(x)) - \int \phi_p(T(x), T(z))d\mu(z) - V_p(T(x))\right\}$$

$$+\left(f^{-1}\right)\prime\left\{-c_{sk}(x, T(x)) - c_{sp}(x, T(x))T^p_k(x) - T^p_k(x)\int \phi_{ps}(T(x), T(z))d\mu(z) - T^p_k(x)V_{sp}(T(x))\right\}$$

that is

$$
\begin{aligned}
\nu_{sr} = \left(f^{-1}\right)'' \times & \\
& \left\{-c_s(x,T(x)) - \int \phi_s(T(x),T(z))d\mu(z) - V_s(T(x))\right\} \\
& \times \left\{-c_p(x,T(x)) - \int \phi_p(T(x),T(z))d\mu(z) - V_p(T(x))\right\} \\
+\left(f^{-1}\right)' & \left\{-c_{sk}(x,T(x)) \left(T^{-1}\right)_r^k - c_{sp}(x,T(x)) - \int \phi_{ps}(T(x),T(z))d\mu(z) - V_{sp}(T(x))\right\}.
\end{aligned}
$$

Now all the terms, with the exception of the $\left(T^{-1}\right)_r^k$ term, are in given by controlled constants, independent of u. We are done. □

Before we state the main a priori estimate, we recall the Ma-Trudinger-Wang tensor [9, pg. 154]. For each y in the target, one can define Ma-Trudinger-Wang (MTW) tensor as a $(2,2)$ tensor on T_xM via

$$\mathsf{MTW}_{ij}^{kl}(x,y) = \left\{\left(-c_{ijpr} + c_{ijs}c^{sm}c_{mrp}\right)c^{pk}c^{rl}\right\}(x,y).$$

It is by now a well known fact that, on the sphere

$$\mathsf{MTW}_{ij}^{kl}\xi_k\xi_l\tau^i\tau^j \geq \delta_n \left\|\xi\right\|^2 \left\|\tau\right\|^2$$

for a positive δ_n and all vector-covector pairs such that

$$\xi(\tau) = 0.$$

(For a more discussion of the geometry of this tensor, see also [5].)

Given a solution, we define an operator on $(2,0)$ tensors as follows. Let h be a $(2,0)$ tensor. Given vector fields X_1, X_2, we define

$$\left(L_w h\right)(X_1,X_2) = \frac{1}{\sqrt{\det w}}\nabla_j\left(\sqrt{\det w}\,w^{ij}\nabla_i h\right) - w^{ij}\nabla_j a \nabla_i h(X_1,X_2)$$

where

$$-a(x) = \frac{1}{2}\ln\det w(x) - \ln\det b(x) + \ln\nu(x,T(x))$$

and covariant differention is taken with respect to the round metric.

Proposition 4. *Let u be a solution of (2.8). If e is a unit direction in a local chart on S^n then*

$$
\begin{aligned}
L_w w(e,e) \geq\ & w^{ij}(-c_{ijpr} + c_{ijs}c_{krp}c^{sk})c^{pm}c^{rl}w_{me}w_{le} \\
& - C\left(1 + \sum w^{ii}\sum w_{jj} + \sum w^{ii} + \sum w_{ii}^2\right)
\end{aligned}
$$

Proof. This was proven by Ma Trudinger and Wang in [9], in the case where densities are known ahead of time. Adapting their proof requires only a small modification somewhere in the middle, but for completeness (and mostly for fun), we will present the calculation.

First, we note that

$$\left(\frac{\partial_j\left(\sqrt{\det w}w^{ij}\right)}{\sqrt{\det w_{ij}}}-w^{ij}a_j\right)=\partial_j w^{ij}+\frac{1}{2}w^{ij}\left(\ln\det w\right)_j+w^{ij}\frac{1}{2}\left(\ln\det w\right)_j-w^{ij}\left(\ln\det b\right)_j+w^{ij}\left(\ln\nu\right)_s T^s_j$$

$$=-w^{ia}w^{bj}\partial_j w_{ab}+w^{ij}\left(\ln\det w\right)_j-w^{ij}\left(b^{sk}b_{skj}+b^{sk}b_{skt}T^t_j\right)+b^{si}\left(\ln\nu\right)_s$$

$$=-w^{ia}w^{bj}(\partial_j w_{ab}-\partial_a w_{bj})-w^{ia}w^{bj}\partial_a w_{bj}+w^{ij}\left(\ln\det w\right)_j-w^{ij}b^{sk}b_{skj}-b^{ti}b^{sk}b_{skt}+b^{si}\left(\ln\nu\right)_s$$

$$=-w^{ia}w^{bj}(c_{abs}T^s_j-c_{bjs}T^s_a)-w^{ij}b^{sk}b_{skj}-b^{ti}b^{sk}b_{skt}+b^{si}\left(\ln\nu\right)_s \tag{4.1}$$

$$=b^{si}w^{bj}c_{bjs}-b^{ti}b^{sk}b_{skt}+b^{si}\left(\ln\nu\right)_s \tag{4.2}$$

using (among others) the relations

$$(\partial_j w_{ab}-\partial_a w_{bj})=c_{abs}T^s_j-c_{bjs}T^s_a$$

$$w^{bj}T^s_j=b^{sj}.$$

Now

$$L_w w(e_1,e_1)=\frac{1}{\sqrt{\det w}}\nabla_j\left(\sqrt{\det w}w^{ij}\nabla_i w\right)(e_1,e_1)-w^{ij}\nabla_j a\nabla_i w(e_1,e_1)$$

$$=w^{ij}\nabla_j\nabla_i w(e_1,e_1)+\left(b^{si}w^{bj}c_{bjs}-b^{ti}b^{sk}b_{skt}+b^{si}\left(\ln\nu\right)_s\right)\nabla_i w(e_1,e_1)$$

$$=w^{ij}\left(\begin{array}{c}\partial_i\partial_j w(e_1,e_1)-\nabla_j\partial_i w(e_1,e_1)+2w(\nabla_{\nabla_j\partial_i}e_1,e_1)\\ 2\partial_i w(\nabla_j e_1,e_1)-2\partial_j w(\nabla_i e_1,e_1)\\ +2w(\nabla_j\nabla_i e_1,e_1)+2w(\nabla_i e_1,\nabla_j e_1)\end{array}\right)$$

$$+\left(b^{si}w^{bj}c_{bjs}-b^{ti}b^{sk}b_{skt}+b^{si}\left(\ln\nu\right)_s\right)\left(\partial_i w(e_1,e_1)-2w(\nabla_i e_1,e_1)\right).$$

At this point, we choose a normal coordinate system (in the round metric), and we have

$$L_w w(e_1,e_1)=\left(b^{si}w^{bj}c_{bjs}-b^{ti}b^{sk}b_{skt}+b^{si}\left(\ln\nu\right)_s\right)\partial_i w(e_1,e_1)+w^{ij}\left(\partial_i\partial_j w(e_1,e_1)+2w(\nabla_j\nabla_i e_1,e_1)\right)$$

$$=\left(b^{is}w^{bj}c_{bjs}-b^{it}b^{sk}b_{skt}+b^{is}\left(\ln\nu\right)_s\right)\partial_i w_{11}$$

$$+w^{ij}\left(\partial_i\partial_j w_{11}-\partial_1\partial_1 w_{ij}\right)+w^{ij}\left(\partial_1\partial_1 w_{ij}+2w(\nabla_j\nabla_i e_1,e_1)\right)$$

Again harking back to [9], we let

$$K=C\sum w^{ii}\sum w_{jj}+C\sum w^{ii}+C\sum w_{ii}^2+C$$

and note that terms of the following form are K

$$K=w^{ij}T^s_b$$

$$K=(\partial_j w_{ik}-\partial_k w_{ij})$$

$$K=w^{ij}2w(\nabla_j\nabla_i e_1,e_1)$$

$$K=w^{ij}w_{kl}$$

so that

$$L_w w(e_1,e_1)=-K+\left(b^{si}w^{bj}c_{bjs}-b^{ti}b^{sk}b_{skt}+b^{si}\left(\ln\nu\right)_s\right)\partial_i w_{11}$$

$$+w^{ij}\left(\partial_i\partial_j w_{11}-\partial_1\partial_1 w_{ij}\right)+w^{ij}\partial_1\partial_1 w_{ij}.$$

Now differentiating

$$\ln\det w_{ij}=\ln\det b_{is}+\ln\mu-\ln\nu \tag{4.3}$$

we have

$$w^{ij}\partial_1 w_{ij} = b^{si}\left(b_{is1} + b_{ist}T_1^t\right) + (\ln\mu)_1 - (\ln\nu)_s T_1^s \tag{4.4}$$

and again

$$w^{ij}\partial_{11}w_{ij} + \partial_1 w^{ij}\partial_1 w_{ij} = K + b^{si}b_{ist}T_{11}^t + (\ln\nu)_{sr}T_1^rT_1^s - (\ln\nu)_s T_{11}^s.$$

Now recall Lemma 3,

$$\begin{aligned}(\ln\nu)_{sr}T_1^rT_1^s &= \frac{C_{1sr} + C_{2sk}\left(T^{-1}\right)_r^k}{\nu}T_1^rT_1^s - (\ln\nu)_s(\ln\nu)_r T_1^rT_1^s\\ &= K\end{aligned}$$

thus

$$w^{ij}\partial_{11}w_{ij} = w^{ia}w^{bj}\partial_1 w_{ab}\partial_1 w_{ij} + K + b^{si}b_{ist}T_{11}^t - (\ln\nu)_s T_{11}^s. \tag{4.5}$$

Note that differentiating

$$T_i^s = b^{sk}w_{ki}$$

yields

$$T_{ij}^s = b^{sk}\partial_j w_{ki} - b^{sa}b^{pk}w_{ki}\left(b_{apj} + b_{apq}T_j^q\right) \tag{4.6}$$

in particular

$$T_{11}^s = b^{sk}\partial_1 w_{k1} - b^{sa}b^{pk}w_{k1}\left(b_{ap1} + b_{apq}T_1^q\right).$$

Now it follows that

$$\begin{aligned}&T_{11}^s - b^{sk}\partial_k w_{11}\\ &= b^{sk}\left(\partial_1 w_{k1} - \partial_k w_{11}\right) - b^{sa}b^{pk}w_{k1}\left(b_{ap1} + b_{apq}T_1^q\right) &(4.7)\\ &= K. &(4.8)\end{aligned}$$

Bringing in concavity of the Monge-Ampère (4.5) and (4.8) we can eliminate some terms to see

$$\begin{aligned}L_w w(e_1,e_1) \geq &-K + b^{is}w^{bj}c_{bjs}\partial_i w_{11}\\ &+ w^{ij}\left(\partial_i\partial_j w_{11} - \partial_1\partial_1 w_{ij}\right).\end{aligned}$$

Then using

$$\begin{aligned}\partial_1\partial_1 w_{ij} &= u_{ij11} + c_{ij11} + 2c_{ijs1}T_1^s + c_{ijs}T_{11}^s + c_{ijpr}T_1^pT_1^r\\ \partial_i\partial_j w_{11} &= u_{11ij} + c_{11ij} + c_{11si}T_j^s + c_{11sj}T_i^s + c_{11s}T_{ij}^s + c_{11pr}T_i^pT_j^r\end{aligned}$$

we have

$$\begin{aligned}L_w w(e_1,e_1) \geq &-K + \left(b^{is}w^{bj}c_{bjs}\right)\partial_i w_{11}\\ &+ w^{ij}\left(c_{11s}T_{ij}^s + c_{11pr}T_i^pT_j^r - c_{ijs}T_{11}^s - c_{ijpr}T_1^pT_1^r\right).\end{aligned}$$

From (4.6)

$$\begin{aligned}w^{ij}T_{ij}^s &= w^{ij}\left(b^{sk}\partial_j w_{ki} - b^{sa}b^{pk}w_{ki}\left(b_{apj} + b_{apq}T_j^q\right)\right)\\ &= w^{ij}b^{sk}\left(\partial_j w_{ki} - \partial_k w_{ij} + \partial_k w_{ij}\right) - b^{sa}b^{pj}\left(b_{apj} + b_{apq}T_j^q\right)\\ &= K + b^{sk}\partial_k\left(\ln\det w\right)\\ &= K\end{aligned}$$

by (4.4). Using (4.7) we conclude

$$L_w w(e_1, e_1) \geq -K - w^{bj} c_{bjs} b^{sa} b^{pk} w_{k1} b_{apq} T_1^q \\ - w^{ij} c_{ijpr} T_1^p T_1^r.$$

which is the desired result after reindexing.

□

Corollary 5. *Second derivatives of u are uniformly bounded.*

Proof. Given the maximum principle estimate, this proof is standard following [9]. For some more details in the setting of Riemannian manifolds see [6, Theorem 3.5]. □

5. Main theorem

In order to make a precise statement, we define

$$\nu_{lower} = f^{-1}\left(f(\frac{1}{n\omega_n}) - 2\mathrm{osc}c - 2\left\|\phi\right\|_\infty - \mathrm{osc}V \right)$$

$$\nu_{upper} = f^{-1}\left(f(\frac{1}{n\omega_n}) + 2\mathrm{osc}c + 2\left\|\phi\right\|_\infty + \mathrm{osc}V \right).$$

Similarly, an upper bound can be defined

$$h_{\max} = \sup_{Q \in [\nu_{lower}, \nu_{upper}]} \frac{\left(f^{-1}(Q)\right)'}{f^{-1}(Q)}.$$

Theorem 6. *Suppose that f satisfies the Inada-like conditions (2.9) (2.10), μ and m are smooth, and ϕ and V are lipschitz. If*

$$\max_{x,y \in M} |\phi(x,y)| < \frac{1}{h_{\max}}. \tag{5.1}$$

then there exists a smooth solution to (3.2).

For existence, we proceed by continuity [4, Theorem 17.6] on the equation (3.2), letting

$$F(t, x, u, Du, D^2u) = \\ \ln\det(\left(D^2 u + D^2 c(x, T(x))\right) - \ln\det(-DDc(x, \bar{T}(x))) \\ - \ln(t\mu(x) + (1-t)m(x)) + \ln f^{-1}\left(Q(t, x, T(x))\right) \tag{5.2}$$

where

$$Q(t, x, T(x)) =$$

$$Q(t, x, T(x)) = -u(x) - c(x, T(x)) - t\int \phi(T(x), T(z))d\mu(z) - tV(T(x))\}.$$

At time $t = 0$, a solution is given by $u \equiv 0$: This maps the measure m to itself via the identity mapping. The interval $\mathcal{I}$ of t for which a solution exists is nonempty. Notice that the form of the equation (5.2) is the same form as (3.2) up to a scale of the functions ϕ and V and a change of measure, so the estimates from the previous section all hold. From the theory of Krylov and Evans one can obtain $C^{2,\alpha}$ estimates. Thus $\mathcal{I}$ is closed. Lemma 2 with these conditions gives openess, noting that on the sphere, a Laplacian has index zero, and that the linearized operator which has the same principle symbol has index zero as well.

Remark. For uniqueness, the standard PDE trick does not work immediately, even under assumptions such as those in the theorem. One may be tempted by the standard argument [4, Theorem 17.1] to obtain a contradiction. However, the intermediate linearized operator will have the additional ∇F term that arises in (3.7) because combinations of u and v are not solutions. Our proof of invertibility fails for these, so we have no reason to expect the proof would hold after being integrated. Uniqueness may be more easily obtained from geometric consideration as in [1, section 4], see also [10, Chapters 15, 16].

However, if the integral term is not present, we can use the argument [4, Theorem 17.1], making the important note that on the sphere, the set of c-convex function is convex [3, Theorem 3.2]. In this case invertibililty of the linearized operator follows easily from standard maximum principle arguments.

References

[1] Adrien Blanchet and Guillaume Carlier. Optimal transport and cournot-nash equilibria. *Preprint*.

[2] Philippe Delanoë and Grégoire Loeper. Gradient estimates for potentials of invertible gradient-mappings on the sphere. *Calc. Var. Partial Differential Equations*, 26(3):297–311, 2006.

[3] Alessio Figalli, Young-Heon Kim, and Robert J. McCann. When is multidimensional screening a convex program? *J. Econom. Theory*, 146(2):454–478, 2011.

[4] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[5] Young-Heon Kim and Robert J. McCann. Continuity, curvature, and the general covariance of optimal transportation. *J. Eur. Math. Soc. (JEMS)*, 12(4):1009–1040, 2010.

[6] Young-Heon Kim, Jeffrey Streets, and Micah Warren. Parabolic optimal transport equations on manifolds. *International Mathematics Research Notices*, 2011.

[7] Jiakun Liu and Neil S. Trudinger. On Pogorelov estimates for Monge-Ampère type equations. *Discrete Contin. Dyn. Syst.*, 28(3):1121–1135, 2010.

[8] Grégoire Loeper. On the regularity of solutions of optimal transportation problems. *Acta Math.*, 202(2):241–283, 2009.

[9] Xi-Nan Ma, Neil S. Trudinger, and Xu-Jia Wang. Regularity of potential functions of the optimal transportation problem. *Arch. Ration. Mech. Anal.*, 177(2):151–183, 2005.

[10] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

Department of Mathematics, Princeton University, Princeton NJ, USA
E-mail address: mww@math.princeton.edu, micahw@uoregon.edu